Algebra with Indefinite Involution and Its Representation in Krein Space

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Abstract

It is often inevitable to introduce an indefinite-metric space in quantum field theory, for example, which is explained for the sake of the manifestly covariant quantization of the electromagnetic field. We show two more evident mathematical reasons why such indefinite metric appears. The first idea is the replacement of involution on an algebra. For an algebra $\mathcal{A}$ with an involution $\dagger$ such that a representation of the involutive algebra $(\mathcal{A},\dagger)$ brings an indefinite-metric space, we replace the involution $\dagger$ with a new one $*$ on $\mathcal{A}$ such that $(\mathcal{A},*)$ is a well-known involutive algebra acting on a representation space with positive definite metric. This explains that non-isomorphic two involutive algebras are transformed each other by the replacement of involution. The second is that a covariant (Hilbert space) representation $(\mathcal{H},\pi,U)$ of an involutive dynamical system $((\mathcal{A},*),\mathbf{Z}_2,\alpha)$ brings a Krein space representation of the algebra $\mathcal{A}$ with the replaced involution. For example, we show representations of abnormal CCRs, CARs and pseudo-Cuntz algebras arising from those of standard CCRs, CARs and Cuntz algebras.

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1 Introduction

We have studied representations of operator algebras [1, 2, 15, 16, 17, 18]. Most of the representation spaces are complex vector spaces with positive definite metric. However, it is often inevitable to introduce an indefinite-metric space in quantum field theory [24, 27]. In this paper, we clarify the mechanism why such indefinite metric appears and show a systematic treatment of such indefinite-metric representation from a standpoint of involutions on algebras.

1.1 Involution and vector space with indefinite metric

We explain terminology here. In this paper, any algebra means an algebra over $\mathbf{C}$. A map φ on $\mathcal{A}$ is called an *involution* on $\mathcal{A}$ if φ is a conjugate linear map which satisfies $\varphi(xy) = \varphi(y)\varphi(x)$ for each $x, y \in \mathcal{A}$ and $\varphi^2 = id$. For convenience, we write x^φ instead of $\varphi(x)$ for $x \in \mathcal{A}$. For the notation of involution, $*, \dagger$ and $\#$ are often used [26, 28, 35]. In physics, for an operator T, the operator $T^\dagger$ is often called the *hermite conjugate* of T, which is the image of an involution $\dagger$ of T. Remark that a different notion of "involution" is used in supersymmetry (§ 5.1.1 in [34]).

The terminology of "$*$-algebra" is not suitable to treat two different involutions on an algebra at once. Hence we use a terminology, involutive algebra instead of it according to Chapter 1 of [6]. A pairing $(\mathcal{A}, *)$ is an *involutive algebra* if $*$ is an involution on an algebra $\mathcal{A}$. Of course, a $*$-algebra $\mathcal{A}$ is an involutive algebra $(\mathcal{A}, *)$. If $x \in \mathcal{A}$ satisfies $x^* = x$, then x is called *$*$-self-adjoint*. An involutive algebra $(\mathcal{A}, *)$ is called a *Banach involutive algebra* if $\mathcal{A}$ is a Banach algebra and $\|x^*\| = \|x\|$ for each $x \in \mathcal{A}$. A Banach involutive algebra $(\mathcal{A}, *)$ is a C^*-algebra if $\|x^*x\| = \|x\|^2$ for each $x \in \mathcal{A}$. For involutive algebras $(\mathcal{A}, *)$ and $(\mathcal{B}, \dagger)$, a homomorphism f from $\mathcal{A}$ to $\mathcal{B}$ is *involutive* if $f \circ * = \dagger \circ f$. An automorphism α of $(\mathcal{A}, *)$ is *involutive* if $* \circ \alpha = \alpha \circ *$. An involution $\dagger$ on $\mathcal{A}$ is *equivalent* to $*$ if there exists an involutive isomorphism from $(\mathcal{A}, \dagger)$ to $(\mathcal{A}, *)$. A subalgebra $\mathcal{A}_0$ of $(\mathcal{A}, *)$ is *involutive* (or *self-adjoint*) if $\{x^* : x \in \mathcal{A}_0\} \subset \mathcal{A}_0$.

An involution is one of most important structures on operator algebras [25]. Especially, the C*-condition is a nice characterization of a special involution with respect to the norm. On the other hand, properties of the involution on the algebra of field operators in quantum field theory is not well-known. For example, the involution on the algebra $\mathcal{A}$ of field operators in quantum electrodynamics satisfies neither the C*-condition nor the positivity of the spectrum of the operator $I + x^*x$ for $x \in \mathcal{A}$.

On a complex vector space V, a map $(\cdot|\cdot)$ from $V \times V$ to $\mathbf{C}$ is called a *hermitian form* on V if $(\cdot|\cdot)$ is sesquilinear and $\overline{(v|w)} = (w|v)$ for each $v, w \in V$. We assume that $(x|cy) = c(x|y)$ for $x, y \in V$ and $c \in \mathbf{C}$ in this paper. In the theory of indefinite-metric space and quantum field theory, such a hermitian form is called by an *inner product* or *metric*. A hermitian form $(\cdot|\cdot)$ is *indefinite* if there exist $v, w \in V$ such that $(w|w) < 0 < (v|v)$. Such pair $(V, (\cdot|\cdot))$ is called an *indefinite-metric space* or *indefinite-inner product space*.

For any operator T on a hermitian vector space $(V, (\cdot|\cdot))$, if $(\cdot|\cdot)$ is nondegenerate, then there exists unique operator $T^\star$ on V such that $(T^\star v|w) = (v|Tw)$ for each $v, w \in V$. The involution $\star$ is called the *involution associated with the hermitian form* $(\cdot|\cdot)$. With respect to this $\star$, the algebra $\mathrm{End}V$ of all linear operators on V is an involutive algebra $(\mathrm{End}V, \star)$. A pairing (V, π) is a *(involutive) representation* of an involutive algebra $(\mathcal{A}, \dagger)$ if V is a vector space with a nondegenerate hermitian form $(\cdot|\cdot)$ and π is an involutive homomorphism from $\mathcal{A}$ to $(\mathrm{End}V, \star)$.

1.2 Abnormal commutation relations

According to the preface in [5], the theory of indefinite-metric space has two origins which are relatively independent. One is quantum field theory [9, 28] and other is functional analysis [30, 31]. A difference of their styles is the order of logic of the theory. In the former, the algebra $\mathcal{A}$ of field operators appears in first and an indefinite-metric space appears as a representation space of $\mathcal{A}$. On the other hand, the later, an indefinite-metric space is given in first and the theory of linear operators on it is discussed [21, 22, 23, 33].

We show that an algebras with a certain involution brings an indefinite-metric space as its involutive representation. In order to explain such algebras, we demonstrate by three examples.

For three families $\{a, a^\dagger\}, \{f, f^\dagger\}$ and $\{s_1, s_2, s_1^\dagger, s_2^\dagger\}$ with an involution $\dagger$, consider the following relations:

$$aa^\dagger - a^\dagger a = -I, \quad aa - aa = a^\dagger a^\dagger - a^\dagger a^\dagger = 0, \tag{1.1}$$

$$ff^\dagger + f^\dagger f = -I, \quad ff + ff = f^\dagger f^\dagger + f^\dagger f^\dagger = 0, \tag{1.2}$$

$$s_i^\dagger s_j = (-1)^{i-1}\delta_{ij} I \quad (i, j = 1, 2), \quad s_1 s_1^\dagger - s_2 s_2^\dagger = I \tag{1.3}$$

where I denotes the unit in each case. Relations (1.1), (1.2) and (1.3) are called the *abnormal canonical commutation relations*, the *abnormal canonical anti-commutation relations*, and the *pseudo-Cuntz relations*, respectively

[2, 27]. Define involutive algebras $\mathcal{A}_{\bar{B}}$, $\mathcal{A}_{\bar{F}}$ and $\mathcal{O}_{1,1}^{(0)}$ generated by them, respectively. We consider their involutive representations as follows.

In Section 3 of [27], it was explained that involutive representations of $\mathcal{A}_{\bar{B}}$ and $\mathcal{A}_{\bar{F}}$ cause indefinite-metric spaces as the involutive representation spaces of them. Assume that $(\cdot|\cdot)$ is a nondegenerate hermitian form on V and $\mathcal{A}_{\bar{B}}$ is involutively represented on $(V,(\cdot|\cdot))$. If there exists a vector $\Omega\in V$ such that $(\Omega|\Omega)>0$ and $a\Omega=0$, then $(a^{\dagger}\Omega|a^{\dagger}\Omega)=-(\Omega|\Omega)<0$. Hence (1.1) brings an indefinite-metric representation in this case. Because the algebra generated by (1.1) is involutively isomorphic to that of canonical commutation relations, the reason why an indefinite-metric space appears may be considered as the choice of the vacuum vector Ω. On the other hand, we can verify that any unital involutive representation of $\mathcal{A}_{\bar{F}}$ must be an indefinite-metric space. In [16], we introduced η-CCRs and η-CARs which are generalization of (1.1) and (1.2), and their representations on Krein spaces by modifying Fock representations of standard CCRs and CARs.

In [2], we introduced $\mathcal{O}_{1,1}^{(0)}$ which is called the *pseudo-Cuntz algebra*, in order to construct representations of the Faddeev-Popov (=FP) (anti) ghost fields in string theory. It is understood that there is no C*-algebra which contains $\mathcal{O}_{1,1}^{(0)}$ as an involutive subalgebra. We construct an involutive representation with indefinite metric of $\mathcal{O}_{1,1}^{(0)}$ as follows. Consider a representation of (V,π) of $\mathcal{O}_{1,1}^{(0)}$ with a cyclic vector Ω such that

$$\pi(s_1)\Omega=\Omega.$$

By (1.3), we see that $\pi(s_2^{\dagger})\Omega=0$. Hence the cyclic representation space V of $\mathcal{O}_{1,1}^{(0)}$ is the linear span of the family $\{\Omega,\pi(s_{j_1}\cdots s_{j_k}s_2)\Omega : j_1,\ldots,j_k = 1,2 \text{ for } k\geq 1\}$ of vectors. Define the hermitian form on V by

$$(e_J|e_K)=(-1)^{n_2(J)}\delta_{JK}$$

where $e_J\equiv\pi(s_{j_1}\cdots s_{j_k})\Omega$ and $n_2(J)\equiv\sum_{i=1}^{k}(j_i-1)$ for $J=(j_1,\ldots,j_k)$. Then $(\cdot|\cdot)$ is nondegenerate on V and $(\Omega|\Omega)=1$. We see that $\mathcal{O}_{1,1}^{(0)}$ acts on $(V,(\cdot|\cdot))$ involutively. Hence $((V,(\cdot|\cdot)),\pi)$ is an involutive representation with indefinite metric of $\mathcal{O}_{1,1}^{(0)}$.

1.3 Replacement of involution

We replace the involution $\dagger$ on $\mathcal{A}_{\bar{B}},\mathcal{A}_{\bar{F}},\mathcal{O}_{1,1}^{(0)}$ by a new one $*$ as follows: Define three automorphisms α, β, γ of $\mathcal{A}_{\bar{B}},\mathcal{A}_{\bar{F}},\mathcal{O}_{1,1}^{(0)}$ by

$$\alpha(a)\equiv -a,\quad \beta(f)\equiv -f,\quad \gamma(s_i)\equiv(-1)^{i-1}s_i\quad (i=1,2). \tag{1.4}$$

Then each of them preserves $\dagger$. Define the new involution $*$ on $\mathcal{A}_{\bar{B}}, \mathcal{A}_{\bar{F}}, \mathcal{O}_{1,1}^{(0)}$ by

$$x^* \equiv \alpha(x^\dagger), \quad y^* \equiv \beta(y^\dagger), \quad z^* \equiv \gamma(z^\dagger) \quad (x \in \mathcal{A}_{\bar{B}},\, y \in \mathcal{A}_{\bar{F}},\, z \in \mathcal{O}_{1,1}^{(0)}).$$

From (1.4), we see that (1.1), (1.2), (1.3) are equivalent to the following equations, respectively:

$$aa^* - a^*a = I, \quad aa - aa = a^*a^* - a^*a^* = 0, \tag{1.5}$$

$$ff^* + f^*f = I, \quad ff + ff = f^*f^* + f^*f^* = 0, \tag{1.6}$$

$$s_i^* s_j = \delta_{ij} I \quad (i, j = 1, 2), \quad s_1 s_1^* + s_2 s_2^* = I. \tag{1.7}$$

New relations (1.5), (1.6) and (1.7) are nothing but CCRs, CARs and the relations of canonical generators of the Cuntz algebra $\mathcal{O}_2$. Let $\mathcal{A}_B$ and $\mathcal{A}_F$ denote algebras generated by $\{a, a^*\}$ and $\{f, f^*\}$, respectively. We see that α, β and γ in (1.4) are also $\mathbf{Z}_2$-actions on $\mathcal{A}_B$, $\mathcal{A}_F$ and $\mathcal{O}_2$, respectively and $x^\dagger = \alpha(x^*)$, $y^\dagger = \beta(y^*)$ and $z^\dagger = \gamma(z^*)$ for $x \in \mathcal{A}_B$, $y \in \mathcal{A}_F$ and $z \in \mathcal{O}_2$.

Furthermore the replacement of involution is not only the change of appearance but also compatible to construct representations of $\mathcal{A}_{\bar{B}}, \mathcal{A}_{\bar{F}}, \mathcal{O}_{1,1}^{(0)}$ on Krein spaces. Remark that $\mathcal{A}_B$ and $\mathcal{A}_F$ are same as $\mathcal{A}_{\bar{B}}$ and $\mathcal{A}_{\bar{F}}$ as algebras if we take no account of their involutions.

Assume that $(\mathcal{H}, \langle\cdot|\cdot\rangle)$ is a Hilbert space and $(\mathcal{H}, \pi, \eta)$ is a covariant representation of the dynamical system $(\mathfrak{A}, \mathbf{Z}_2, \varphi) = (\mathcal{A}_B, \mathbf{Z}_2, \alpha), (\mathcal{A}_F, \mathbf{Z}_2, \beta)$, $(\mathcal{O}_2, \mathbf{Z}_2, \gamma)$, that is, $\pi \circ \varphi = \mathrm{Ad}\eta \circ \pi$. Define $(\cdot|\cdot) \equiv \langle\cdot|\eta(\cdot)\rangle$. Then we see that η is a self-adjoint unitary on $\mathcal{H}$ and $(\mathcal{H}, (\cdot|\cdot))$ is a Krein space such that

$$(\pi(x^\dagger)v|w) = (v|\pi(x)w) \quad (v, w \in \mathcal{H},\, x \in \mathfrak{A}).$$

Define $\mathfrak{A}_\pm \equiv \{x \in \mathfrak{A} : \varphi(x) = \pm x\}$. Then

$$\pi(\mathfrak{A}_+)\mathcal{H}_\pm \subset \mathcal{H}_\pm, \quad \pi(\mathfrak{A}_-)\mathcal{H}_\pm \subset \mathcal{H}_\mp \tag{1.8}$$

where $\mathcal{H}_\pm \equiv \{v \in \mathcal{H} : \eta v = \pm v\}$. In this way, we see that a covariant representation of the involutive algebra $(\mathfrak{A}, *)$ is closely related to the Krein space representation of another involutive algebra $(\mathfrak{A}, \dagger)$.

In Section 2, we will introduce indefinite involutions in order to show the difference between $\dagger$ and $*$ in the above three examples and define Krein C*-algebras which are generalizations of C*-algebras including $\mathcal{A}_{\bar{F}}, \mathcal{O}_{1,1}^{(0)}$ and the algebra of the Weyl form of abnormal CCRs. In Sections 3, 4 and 5, we will introduce elementary examples, the η-CCR algebras, the η-CAR algebras and the pseudo-Cuntz algebras as examples of Krein C*-algebra. In Appendix, we will show two models of indefinite-metric quantum field theory.

2 Krein C*-algebra

We introduce indefinite involutions and Krein C*-algebras in this section.

2.1 Krein space and indefinite involution on algebra

A hermitian vector space $(\mathcal{V}, (\cdot|\cdot))$ is a *Krein space* if there exists a decomposition $\mathcal{V} = \mathcal{V}_+ \oplus \mathcal{V}_-$ and $(\mathcal{V}_\pm, \pm(\cdot|\cdot))$ is a Hilbert space [4, 5, 12]. This decomposition is called a *fundamental decomposition* of $(\mathcal{V}, (\cdot|\cdot))$. By definition, the new hermitian form $\langle\cdot|\cdot\rangle$ on $\mathcal{H}$ defined by $\langle v|w\rangle \equiv (v|E_+w) - (v|E_-w)$ for $v, w \in \mathcal{H}$, is positive definite where $E_\pm$ denotes the projection from $\mathcal{H}$ onto $\mathcal{H}_\pm$. The operator $U \equiv E_+ - E_-$ is called a *fundamental symmetry* of $(\mathcal{H}, (\cdot|\cdot))$. For a given Krein space, its fundamental decomposition is not unique in general. Hence we use a *Krein triplet* $(\mathcal{H}, \langle\cdot|\cdot\rangle, \eta)$, that is, a Hilbert space $(\mathcal{H}, \langle\cdot|\cdot\rangle)$ and a self-adjoint unitary η on $\mathcal{H}$. For a Krein triplet $(\mathcal{H}, \langle\cdot|\cdot\rangle, \eta)$, let $\mathcal{H}_\pm \equiv \{v \in \mathcal{H} : \eta v = \pm v\}$. Then $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. Hence $(\mathcal{H}, (\cdot|\cdot))$ is a Krein space with the nondegenerate hermitian form $(\cdot|\cdot)$ defined by $(v|w) \equiv \langle v|\eta w\rangle$ for $v, w \in \mathcal{H}$.

For an algebra $\mathcal{A}$ with a unit I, the *spectrum* $\mathrm{sp}_{\mathcal{A}}(a)$ of $a \in \mathcal{A}$ is defined by the subset $\{z \in \mathbf{C} : \text{there exists no inverse of } A - \lambda I \text{ in } \mathcal{A}\}$ of $\mathbf{C}$ [6]. We write $\mathrm{sp}_{\mathcal{A}}(a)$ as $\mathrm{sp}(a)$ for simplicity of description. Remark that the definition of the spectrum is written without use of any topology. Important results of spectrum theory do not hold without use of a norm, especially, the C*-condition. However, there is no assumption of the existence of a norm on the algebra of field operators in quantum field theory in general. Therefore it is necessary to characterize a suitable assumption for the involution on such an algebra without topology.

Definition 1 *Let $(\mathcal{A}, *)$ be an involutive algebra with a unit I.*

(i) *The involution $*$ is positive definite if $I + x^*x$ is invertible for each $x \in \mathcal{A}$.*

(ii) *The involution $*$ is indefinite if there exist $x, y \in \mathcal{A}$ such that* $\mathrm{sp}(x^*x) \cap (0, \infty) \neq \emptyset$ *and* $\mathrm{sp}(y^*y) \cap (-\infty, 0) \neq \emptyset$.

The involution on any C*-algebra is positive definite [11, 14, 19]. If $\mathcal{A}$ has a unit I, then $I^* = I$ for each involution $*$ on $\mathcal{A}$. Hence if there exists $x \in \mathcal{A}$ such that $\mathrm{sp}(x^*x) \cap (-\infty, 0) \neq \emptyset$, then $*$ is indefinite. By definition, if $*$ is positive definite, then $\mathrm{sp}(x^*x)$ is a subset of $[0, \infty)$ for each $x \in \mathcal{A}$.

2.2 Krein C*-algebra

We introduce Krein C*-algebras in this subsection. For an involutive algebra $(\mathcal{A},*)$, we write $\mathrm{Aut}(\mathcal{A},*)$ the set of all involutive automorphisms of $(\mathcal{A},*)$ and define $\mathrm{Aut}_2(\mathcal{A},*) \equiv \{\alpha \in \mathrm{Aut}(\mathcal{A},*) : \alpha^2 = id\}$. For any $\alpha \in \mathrm{Aut}_2(\mathcal{A},*)$, $\alpha \circ *$ is also an involution on $\mathcal{A}$. If $(\mathcal{A},*)$ is a unital C*-algebra and $\alpha \neq id$, then the involution $\dagger$ defined by $x^\dagger \equiv \alpha(x^*)$ is indefinite.

We generalize the notion of C*-algebra according to the definition of Krein space.

Definition 2 *A Banach involutive algebra $(\mathcal{A},\dagger)$ is called a Krein C*-algebra if there exists $\alpha \in \mathrm{Aut}_2(\mathcal{A},\dagger)$ such that*

$$\|\alpha(x^\dagger)x\| = \|x\|^2 \quad \textit{for all } x \in \mathcal{A}.$$

In this case, α is called a fundamental symmetry of $(\mathcal{A},\dagger)$.

By definition, if $(\mathcal{A},\dagger)$ is a Krein C*-algebra with a fundamental symmetry α, then $(\mathcal{A},\alpha\circ\dagger)$ is a C*-algebra and $* \equiv \alpha \circ \dagger$ satisfies $* \circ \alpha = \alpha \circ *$. We do not know that the uniqueness of fundamental symmetry of a given Krein C*-algebra $(\mathcal{A},\dagger)$. Hence we define a *Krein triplet* $(\mathcal{A},*,\alpha)$ by a C*-algebra $(\mathcal{A},*)$ and $\alpha \in \mathrm{Aut}_2(\mathcal{A},*)$. This is nothing but a C*-dynamical system $((\mathcal{A},*),\mathbf{Z}_2,\alpha)$.

For two Krein triplets $(\mathcal{A},*,\alpha)$ and $(\mathcal{B},\dagger,\beta)$ of C*-algebras are *isomorphic* if there exists an involutive isomorphism ψ from $(\mathcal{A},*)$ and $(\mathcal{B},\dagger)$ such that $\psi\circ\alpha = \beta\circ\psi$. An algebra $\mathcal{B}$ is a *subalgebra* of Krein triplet $(\mathcal{A},*,\alpha)$ of C*-algebra if $\mathcal{B}$ is a C*-subalgebra of $\mathcal{A}$ such that $\alpha(\mathcal{B}) = \mathcal{B}$. If $(\mathcal{A},*,\alpha)$ and $(\mathcal{B},\dagger,\beta)$ are isomorphic, then $(\mathcal{A},\#,\alpha)$ and $(\mathcal{B},\star,\beta)$ are isomorphic where $x^\# \equiv \alpha(x^*)$ and $y^\star \equiv \beta(y^\dagger)$ for $x \in \mathcal{A}$ and $y \in \mathcal{B}$.

If $(\mathcal{A},*,\alpha)$ is a Krein triplet of C*-algebra, then there is the following natural decomposition as a Banach space:

$$\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_-, \quad \mathcal{A}_\pm \equiv \{x \in \mathcal{A} : \alpha(x) = \pm x\}. \tag{2.1}$$

We see that $\{x^* : x \in \mathcal{A}_\pm\} \subset \mathcal{A}_\pm$. Especially, $\mathcal{A}_+$ is the fixed-point subalgebra $\mathcal{A}^\alpha$ of $\mathcal{A}$ with respect to α. Define $x^\dagger \equiv \alpha(x^*)$ for $x \in \mathcal{A}$. Then

$$x^\dagger x = \pm x^* x \quad (x \in \mathcal{A}_\pm).$$

In this sense, $\dagger$ on $\mathcal{A}_+$ (*resp.* $\mathcal{A}_-$) is positive definite (*resp.* negative definite) because $x^*x \geq 0$ for each $x \in \mathcal{A}$. Therefore the decomposition in (2.1) is regarded as an analogy of a fundamental decomposition of a Krein space.

Because $x^* = \alpha(x^*)$ for each $x \in \mathcal{A}_+$, the Banach involutive algebra $(\mathcal{A}_+, \dagger)$ is a C*-algebra. Rewrite $X_\alpha \equiv \mathcal{A}_-$. Then $(X_\alpha, \langle\cdot|\cdot\rangle)$ is a Hilbert $\mathcal{A}_+$-module where $\langle a|b\rangle \equiv a^*b$ for $a, b \in X_\alpha$. If $x \in \mathcal{A}$ is $\dagger$-self-adjoint, then there are unique $x_\pm \in \mathcal{A}_\pm$ such that $x_\pm^* = x_\pm$ and $x = x_+ + \sqrt{-1}x_-$. Hence

$$\mathcal{A}_{\dagger s.a} = (\mathcal{A}_+)_{*s.a} \oplus \sqrt{-1}(\mathcal{A}_-)_{*s.a}$$

where $\mathcal{A}_{\dagger s.a}$ denotes the set of all $\dagger$-self-adjoint elements in $\mathcal{A}$ and others are defined as the same way. The part of $\sqrt{-1}(\mathcal{A}_-)_{*s.a}$ often brings imaginary spectra of $\dagger$-self-adjoint operators in quantum field theory.

We show the similarity between Krein spaces and Krein C*-algebras as follows:

	Krein space $\mathcal{V}$	Krein C*-algebra $\mathcal{A}$
definition	Hilbert space with a unitary η, $\eta^2 = I$	C*-algebra with an automorphism α, $\alpha^2 = id$
structure	indefinite metric $(\cdot|\cdot)$	indefinite involution $\dagger$
fundamental symmetry	η	α
fundamental decomposition	$\mathcal{V} = \mathcal{V}_+ \oplus \mathcal{V}_-$	$\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_-$
positive definite object	Hilbert space	C*-algebra

2.3 Construction of involutive representation of Krein C*-algebra

We construct an involutive representation of an algebra with indefinite involution from an algebra with positive definite involution.

Definition 3 *A linear map T on a Krein space $(\mathcal{H}, (\cdot|\cdot))$ is bounded if T is bounded with respect to the standard Hilbert space of $(\mathcal{H}, (\cdot|\cdot))$. We write $\mathcal{B}(\mathcal{H})$ the set of all bounded linear operators on $\mathcal{H}$.*

A triplet $(\mathcal{A}, G, \alpha)$ is a *C*-dynamical system* if α is a continuous action of the (topological) group G on a C*-algebra $\mathcal{A}$. Especially, if $\alpha \in \mathrm{Aut}\mathcal{A}$ satisfies $\alpha^2 = id$, then we obtain a C*-dynamical system $(\mathcal{A}, \mathbf{Z}_2, \alpha)$ associated with α.

Theorem 1 *Let $(\mathcal{A}, *, \alpha)$ be a Krein triplet of C*-algebra and define the involution $\dagger$ on $\mathcal{A}$ by $x^\dagger \equiv \alpha(x^*)$ for $x \in \mathcal{A}$. Assume that $\mathcal{H}$ is a Hilbert space*

with the positive definite metric $\langle\cdot|\cdot\rangle$ and $(\mathcal{H},\pi,\eta)$ is a covariant representation of the C^*-dynamical system $((\mathcal{A},*),\mathbf{Z}_2,\alpha)$. Define the hermitian form $(\cdot|\cdot)$ on $\mathcal{H}$ by

$$(v|w) \equiv \langle v|\eta w\rangle \quad (v,w\in\mathcal{H}). \tag{2.2}$$

Then $\mathcal{H}$ is a Krein space with respect to the hermitian form $(\cdot|\cdot)$ and π is an involutive representation of $(\mathcal{A},\dagger)$ on $(\mathcal{H},(\cdot|\cdot))$. Furthermore the following holds:

$$\pi(x)^\star = \eta\pi(x^*)\eta \quad (x\in\mathcal{A})$$

where $\star$ denotes the involution associated with the hermitian form $(\cdot|\cdot)$.

Proof. Since $\pi\circ\alpha = \mathrm{Ad}\eta\circ\pi$, we see that

$$\begin{aligned}\langle\pi(x)^\star v|w\rangle =&\ (\pi(x)^\star v|\eta w)\\ =&\ (v|\pi(x)\eta w)\\ =&\ (v|\eta\pi(\alpha(x))w)\\ =&\ \langle v|\pi(\alpha(x))w\rangle\\ =&\ \langle\pi(\alpha(x))^* v|w\rangle\\ =&\ \langle\pi(\alpha(x)^*)v|w\rangle\\ =&\ \langle\pi(x^\dagger)v|w\rangle\end{aligned}$$

for each $x\in\mathcal{A}$ and $v,w\in\mathcal{H}$. From this, $\pi(x)^\star = \pi(x^\dagger)$ for each $x\in\mathcal{A}$. Hence the former statement holds. The later is verified by the definition of $\dagger$. ∎

Under the same assumption in Theorem 1, we see that

$$\pi(\mathcal{A}_+)\mathcal{H}_\pm \subset \mathcal{H}_\pm, \quad \pi(\mathcal{A}_-)\mathcal{H}_\pm \subset \mathcal{H}_\mp \tag{2.3}$$

where $\mathcal{H}_\pm \equiv \{v\in\mathcal{H} : \eta v = \pm v\}$ and $\mathcal{A}_\pm$ is as in (2.1). Furthermore $\pi(\mathcal{A})\cap\mathcal{B}(\mathcal{H})_\pm = \pi(\mathcal{A}_\pm)$ where $\mathcal{B}(\mathcal{H})_+ \equiv \{a\in\mathcal{B}(\mathcal{H}) : a\mathcal{H}_\pm\subset\mathcal{H}_\pm\}$ and $\mathcal{B}(\mathcal{H})_- \equiv \{a\in\mathcal{B}(\mathcal{H}) : a\mathcal{H}_\pm\subset\mathcal{H}_\mp\}$. In this way, the decomposition in (2.1) is compatible with the fundamental decomposition of the Krein space $(\mathcal{H},(\cdot|\cdot))$. From (2.3) and $* = \dagger$ on $\mathcal{A}_+$, the Hilbert subspace $(\mathcal{H}_+,(\cdot|\cdot))$ is an involutive representation of the involutive subalgebra $\mathcal{A}_+$ of $(\mathcal{A},\dagger)$.

By Theorem 1, the following holds.

Corollary 1 *Let $(\mathcal{A},*,\alpha)$ be a Krein triplet of C^*-algebra. Assume that ω is a state on $(\mathcal{A},*)$ such that the GNS representation of $\mathcal{A}$ by $\omega\circ\alpha$ is equivalent to that by ω. Then there exists a self-adjoint unitary η on $\mathcal{H}$ for the GNS representation $(\mathcal{H},\pi)$ by ω such that $(\mathcal{H},\pi,(\cdot|\cdot))$ is an involutive representation of $(\mathcal{A},\dagger)$ with respect to $(\cdot|\cdot)$ in (2.2).*

For a representation $(\mathcal{H},\pi)$ of a C*-algebra $\mathcal{A}$ and a C*-dynamical system $(\mathcal{A},G,\alpha)$ with a locally compact group G, the *regular representation* $(L_2(G,\mathcal{H}),\tilde{\pi}\rtimes\lambda)$ of the crossed product $\mathcal{A}\rtimes G$ by $(\mathcal{H},\pi)$ is the representation which is induced by the following covariant representation $(L_2(G,\mathcal{H}),\tilde{\pi},\lambda)$ as follows (Section 7.7, [29]):

$$(\tilde{\pi}(a)\phi)(g) \equiv \pi(\alpha_{g^{-1}}(a))\phi(g), \quad (\lambda_h\phi)(g) \equiv \phi(h^{-1}g) \tag{2.4}$$

for $a \in \mathcal{A}$, $g,h \in G$ and $\phi \in L_2(G,\mathcal{H})$ where $L_2(G,\mathcal{H})$ denotes the Hilbert space of all square integrable $\mathcal{H}$-valued functions on G with respect to the Haar measure of G. We apply this for the finite group $G = \mathbf{Z}_2$. Let $(\mathcal{H},\pi)$ be a representation of a C*-algebra $\mathcal{A}$ and let α be an action of $\mathbf{Z}_2$ on $\mathcal{A}$. Assume that there is no unitary U on $\mathcal{H}$ such that $\mathrm{Ad}U \circ \pi = \pi \circ \alpha$. Define the representation $\tilde{\pi}$ of $\mathcal{A}$ on the Hilbert space $\tilde{\mathcal{H}} \equiv \mathcal{H} \otimes \mathbf{C}^2$ by

$$\tilde{\pi}(x)(v \otimes e_i) \equiv \{\pi(\alpha^i(x))v\} \otimes e_i \quad (v \in \mathcal{H},\, x \in \mathcal{A},\, i = 0,1) \tag{2.5}$$

where e_0 and e_1 are standard basis of $\mathbf{C}^2$. Define the unitary η on $\tilde{\mathcal{H}}$ by

$$\eta v \otimes e_i \equiv v \otimes e_{1-i} \quad (v \in \mathcal{H},\, i = 0,1).$$

Then we can verify that $\eta\tilde{\pi}(x)\eta^* = \tilde{\pi}(\alpha(x))$ for $x \in \mathcal{A}$. Hence $(\tilde{\mathcal{H}},\tilde{\pi},\eta)$ is a covariant representation of the C*-dynamical system $(\mathcal{A},\mathbf{Z}_2,\alpha)$. Define the hermitian form $(\cdot|\cdot)$ on $\tilde{\mathcal{H}}$ by $(\cdot|\cdot) \equiv \langle\cdot|\eta(\cdot)\rangle$ and $\tilde{\mathcal{H}}_\pm \equiv \{z \in \tilde{\mathcal{H}} : \eta z = \pm z\}$. Then

$$\tilde{\mathcal{H}}_+ = \{v \otimes (e_0 + e_1) : v \in \mathcal{H}\}, \quad \tilde{\mathcal{H}}_- = \{v \otimes (e_0 - e_1) : v \in \mathcal{H}\}.$$

In this way, we can always construct an involutive representation which satisfies the assumption in Theorem 1 from a given involutive representation of a C*-algebra with a $\mathbf{Z}_2$-action.

3 Elementary examples

We show elementary examples of Theorem 1.

Example 1 Let $C[0,1]$ denote the unital commutative C*-algebra of all complex-valued continuous functions on the interval $[0,1]$ with respect to the standard operations. Define the new involution $\dagger$ on $C[0,1]$ by

$$f^\dagger(x) \equiv \overline{f(1-x)} \quad (f \in C[0,1],\, x \in [0,1]).$$

Define $f_1 \in C[0,1]$ by $f_1(x) \equiv 1-2x$. Then $\{f_1^\dagger f_1\}(x) = -(1-2x)^2 \le 0$. Therefore $\dagger$ is an indefinite involution on $C[0,1]$. Define $\alpha \in \mathrm{Aut}C[0,1]$ by $\alpha(f)(x) \equiv f(1-x)$. Then $f^\dagger = \alpha(\bar{f})$. Define the representation π of $C[0,1]$ on the Hilbert space $(L_2[0,1], \langle\cdot|\cdot\rangle)$ by

$$\{\pi(f)\phi\}(x) \equiv f(x)\phi(x) \quad (f \in C[0,1],\, \phi \in L_2[0,1],\, x \in [0,1]).$$

Define the self-adjoint unitary η on $L_2[0,1]$ by

$$(\eta\phi)(x) \equiv \phi(1-x) \quad (\phi \in L_2[0,1],\, x \in [0,1]).$$

Then $(L_2[0,1], \pi, \eta)$ is a covariant representation of the C*-dynamical system $(C[0,1], \mathbf{Z}_2, \alpha)$. Hence we obtain the involutive representation π of the Krein C*-algebra $(C[0,1], \dagger)$ on the Krein space $(L_2[0,1], (\cdot|\cdot))$ where $(\cdot|\cdot) \equiv \langle\cdot|\eta(\cdot)\rangle$.

Example 2 (Involutions on $M_2(\mathbf{C})$) The importance of involutions on matrix algebras are well-known according to Weyl's unitary trick [36, 37, 38]. We consider relations between such involutions and indefinite involutions as follows. Let $M_2(\mathbf{C})$ denote the unital C*-algebra of all 2×2 matrices with complex entries with respect to the standard operations. We consider involutions on $M_2(\mathbf{C})$ associated with Pauli's spin matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let σ_0 denote the identity matrix for convenience. Define

$$\alpha_i \equiv \mathrm{Ad}\sigma_i \in \mathrm{Aut}M_2(\mathbf{C}), \quad x^{\dagger_i} \equiv \alpha_i(x^*) \quad (x \in M_2(\mathbf{C}),\, i = 0,1,2,3).$$

We rewrite $\mathfrak{gl}(2,\mathbf{C}) = M_2(\mathbf{C})$ as a complex Lie algebra with the Lie bracket of the standard commutator. Define the family $\{\mathfrak{u}_{\dagger_i}(2) : i = 0,1,2,3\}$ of real Lie subalgebras of $\mathfrak{gl}(2,\mathbf{C})$ by

$$\mathfrak{u}_{\dagger_i}(2) \equiv \{X \in \mathfrak{gl}(2,\mathbf{C}) : X^{\dagger_i} + X = 0\} \quad (i = 0,1,2,3).$$

Then $x^{\dagger_0}$ is the standard hermitian conjugate of $x \in M_2(\mathbf{C})$ and $\mathfrak{u}_{\dagger_0}(2)$ is the Lie algebra $\mathfrak{u}(2)$ of $U(2)$. Let $* \equiv \dagger_0$. Then we can verify that any two of involutive algebras $(M_2(\mathbf{C}), \dagger_1)$, $(M_2(\mathbf{C}), \dagger_2)$ and $(M_2(\mathbf{C}), \dagger_3)$ are involutively isomorphic. Since such isomorphisms can be chosen as preserving the trace, $\mathfrak{u}_{\dagger_1}(2), \mathfrak{u}_{\dagger_2}(2)$ and $\mathfrak{u}_{\dagger_3}(2)$ are also involutively isomorphic as a Lie algebra. We explain more concretely as follows: For $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$\alpha_1(X) = \begin{pmatrix} d & c \\ b & a \end{pmatrix}, \quad \alpha_2(X) = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}, \quad \alpha_3(X) = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}.$$

Hence

$$X^{\dagger_1} = \begin{pmatrix} \bar{d} & \bar{b} \\ \bar{c} & \bar{a} \end{pmatrix}, \quad X^{\dagger_2} = \begin{pmatrix} \bar{d} & -\bar{b} \\ -\bar{c} & \bar{a} \end{pmatrix}, \quad X^{\dagger_3} = \begin{pmatrix} \bar{a} & -\bar{c} \\ -\bar{b} & \bar{d} \end{pmatrix}.$$

From these,

$$\begin{aligned} \mathfrak{u}_{\dagger_1}(2) = & \left\{ \begin{pmatrix} a & \sqrt{-1}b \\ \sqrt{-1}c & -\bar{a} \end{pmatrix} : a \in \mathbf{C},\, b, c \in \mathbf{R} \right\}, \\ \mathfrak{u}_{\dagger_2}(2) = & \ \mathfrak{gl}(2, \mathbf{R}), \\ \mathfrak{u}_{\dagger_3}(2) = & \ \{X \in \mathfrak{gl}(2, \mathbf{C}) : I_{1,1}X + X^* I_{1,1} = 0\} = \mathfrak{u}(1,1) \end{aligned}$$

where $I_{1,1} = \sigma_3$. We see that $I_{1,1}^{\dagger_1} I_{1,1} = -I$. Therefore $\dagger_1$ is an indefinite involution on $M_2(\mathbf{C})$. This implies that both $\dagger_2$ and $\dagger_3$ are also indefinite and neither $\dagger_1$, $\dagger_2$ nor $\dagger_3$ is involutively isomorphic to $\dagger_0$ because $\dagger_0$ is positive definite. For any $X \in \mathfrak{u}_{\dagger_i}(2)$ $(i = 1, 2, 3)$, we see that $\exp X$ is not unitary. However the definition of $\mathfrak{u}_{\dagger_i}(2)$ is same as $\mathfrak{u}(2)$ if we misunderstand $\dagger_i$ as $*$ for each $i = 1, 2, 3$.

These elementary examples reveal a question about quantum field theory. In quantum field theory, a symmetry is described as a unitary (=anti-hermitian) representation of a Lie algebra $\mathfrak{g}$. Such assumption is formulated as

$$X^\dagger + X = 0$$

for a generator X of $\mathfrak{g}$ by using a certain involution $\dagger$. However there exists no assumption of the positive definiteness for $\dagger$ because the metric of the representation of the theory is unknown until one computes expectation values of field operators in general. Therefore we can not know whether $\dagger$ is positive definite or not.

Examples 1 and 2 show that neither the non-commutativity nor the infinite-dimensionality of an algebra is essential for indefinite involution. The involution $\dagger$ on the algebra generated by each relation (1.1), (1.2) and (1.3) is indefinite.

Example 3 Let $(\mathcal{H}, \langle\cdot|\cdot\rangle, \eta)$ be a Krein triplet and let $\mathcal{B}(\mathcal{H})$ denote the C*-algebra of all bounded linear operators on the Hilbert space $(\mathcal{H}, \langle\cdot|\cdot\rangle)$. For $x \in \mathcal{B}(\mathcal{H})$, the adjoint $x^\dagger$ of x with respect to the hermitian form $(\cdot|\cdot) \equiv \langle\cdot|\eta(\cdot)\rangle$ satisfies

$$x^\dagger = \eta x^* \eta^*.$$

If $\eta \neq I$, then $\dagger$ is an indefinite involution on $\mathcal{B}(\mathcal{H})$ because $(\cdot|\cdot) \equiv \langle\cdot|\eta(\cdot)\rangle$ is positive definite if and only if $\dagger$ is positive definite.

4 η-CCR and η-CAR algebra

In [16], we introduced η-CCRs and η-CARs as families of operators on Krein spaces. Here we reformulate them without use of representation. For a Hilbert space $(\mathcal{H}, \langle\cdot|\cdot\rangle)$, let $(\mathfrak{A}_B(\mathcal{H}), *)$ and $(\mathfrak{A}_F(\mathcal{H}), *)$ denote the CCR algebra and the CAR algebra over $\mathcal{H}$, respectively (Section 5.2.1 in [7]). Define $(\mathcal{A}(\mathcal{H}), *)$ the involutive algebra generated by CCRs $\{a(f), a^*(f) : f \in \mathcal{H}\}$ over $\mathcal{H}$. Note that $(\mathfrak{A}_B(\mathcal{H}), *)$ and $(\mathfrak{A}_F(\mathcal{H}), *)$ are unital C*-algebras but $(\mathcal{A}(\mathcal{H}), *)$ is not. Let η be a self-adjoint unitary on $\mathcal{H}$.

4.1 η-CCR algebra

Let $(\mathfrak{A}_B^{(0)}(\mathcal{H}, \eta), \dagger)$ denote the involutive algebra generated by a family $\{W(f) : f \in \mathcal{H}\}$ which satisfies

$$\{W(f)\}^\dagger = W(-\eta f), \quad W(f)W(g) = e^{-\sqrt{-1}\mathrm{Im}\langle f|g\rangle/2}W(f+g) \quad (f, g \in \mathcal{H}).$$

Define the involutive automorphism α on $(\mathfrak{A}_B^{(0)}(\mathcal{H}, \eta), \dagger)$ by $\alpha(W(f)) \equiv W(\eta f)$ for $f \in \mathcal{H}$.

Lemma 1 *There exists a unique norm $\|\cdot\|$ on $\mathfrak{A}_B^{(0)}(\mathcal{H}, \eta)$ such that $\|\alpha(x^\dagger)x\| = \|x\|^2$ for each $x \in \mathfrak{A}_B^{(0)}(\mathcal{H}, \eta)$.*

Proof. Define the new involution $*$ on $\mathfrak{A}_B^{(0)}(\mathcal{H}, \eta)$ by $x^* \equiv \alpha(x^\dagger)$ for $x \in \mathfrak{A}_B^{(0)}(\mathcal{H}, \eta)$. Then we see that $\{W(f) : f \in \mathcal{H}\}$ satisfies the canonical relations of the Weyl form of CCRs with respect to the new involution $*$. Therefore the involutive algebra $(\mathfrak{A}_B^{(0)}(\mathcal{H}, \eta), *)$ is densely embedded into the CCR algebra $\mathfrak{A}_B(\mathcal{H})$. On the other hand, the assumption of the norm in the statement is just the C*-norm on $(\mathfrak{A}_B^{(0)}(\mathcal{H}, \eta), *)$. By the uniqueness of the C*-norm on $\mathfrak{A}_B(\mathcal{H})$, the statement holds. ∎

Definition 4 *The completion $\mathfrak{A}_B(\mathcal{H}, \eta)$ of $\mathfrak{A}_B^{(0)}(\mathcal{H}, \eta)$ with respect to the norm in Lemma 1 is called the η-CCR algebra over $\mathcal{H}$.*

The algebra $\mathfrak{A}_B(\mathcal{H}, \eta)$ is a Krein C*-algebra with a fundamental symmetry α.

Let $(\mathcal{A}_B(\mathcal{H},\eta),\dagger)$ denote the involutive algebra generated by a family $\{a(f),a^\dagger(f):f\in\mathcal{H}\}$ which satisfies

$$\begin{cases} \{a(f)\}^\dagger = a^\dagger(f), & \\ a(f)a^\dagger(g) - a^\dagger(g)a(f) = \langle f|\eta g\rangle I, & (f,g\in\mathcal{H}). \\ a(f)a(g) - a(g)a(f) = a^\dagger(f)a^\dagger(g) - a^\dagger(g)a^\dagger(f) = 0 & \end{cases}$$

We call $(\mathcal{A}_B(\mathcal{H},\eta),\dagger)$ the *algebra of CCRs* over $\mathcal{H}$. Similar algebras are treated in Section 4 of [20].

4.2 η-CAR algebra

Let $(\mathfrak{A}_F^{(0)}(\mathcal{H},\eta),\dagger)$ denote the involutive algebra generated by a family $\{a(f),a^\dagger(f):f\in\mathcal{H}\}$ which satisfies

$$\begin{cases} \{a(f)\}^\dagger = a^\dagger(f), & \\ a(f)a^\dagger(g) + a^\dagger(g)a(f) = \langle f|\eta g\rangle I, & (f,g\in\mathcal{H}). \\ a(f)a(g) + a(g)a(f) = a^\dagger(f)a^\dagger(g) + a^\dagger(g)a^\dagger(f) = 0 & \end{cases}$$

Define the involutive automorphism α on $(\mathfrak{A}_F^{(0)}(\mathcal{H},\eta),\dagger)$ by $\alpha(a(f))\equiv a(\eta f)$ for $f\in\mathcal{H}$.

Lemma 2 *There exists a unique norm $\|\cdot\|$ on $\mathfrak{A}_F^{(0)}(\mathcal{H},\eta)$ such that $\|\alpha(x^\dagger)x\| = \|x\|^2$ for each $x\in\mathfrak{A}_F^{(0)}(\mathcal{H},\eta)$.*

Proof. In the similarity of the proof of Lemma 1, the statement holds. ∎

Definition 5 *The completion $\mathfrak{A}_F(\mathcal{H},\eta)$ of $\mathfrak{A}_F^{(0)}(\mathcal{H},\eta)$ with respect to the norm in Lemma 2 is called the η-CAR algebra over $\mathcal{H}$.*

The algebra $\mathfrak{A}_F(\mathcal{H},\eta)$ is also a Krein C*-algebra with a fundamental symmetry α.

4.3 Representation in Krein space

In [16], we have already given involutive representations (= the η-Fock representations) of $(\mathcal{A}_B(\mathcal{H},\eta),\dagger)$ and $(\mathfrak{A}_F(\mathcal{H},\eta),\dagger)$.

Theorem 2 *Let $(\mathcal{H}, \langle\cdot|\cdot\rangle, \eta)$ be a Krein triplet and let $\mathcal{F}_+(\mathcal{H})$ and $\mathcal{F}_-(\mathcal{H})$ denote the completely symmetric and the completely anti-symmetric Fock space over the Hilbert space $(\mathcal{H}, \langle\cdot|\cdot\rangle)$, respectively. Let Ω denote their vacuum vectors as the same symbol.*

(i) *There exists a self-adjoint unitary $\Gamma(\eta)$ on $\mathcal{F}_+(\mathcal{H})$ and an involutive representation π_B of $(\mathfrak{A}_B(\mathcal{H},\eta), \dagger)$ on the Krein space $(\mathcal{F}_+(\mathcal{H}), (\cdot|\cdot))$ such that*
$$(\Omega|\pi_B(W(f))\Omega) = e^{-\|f\|^2/4} \quad (f \in \mathcal{H})$$
where $(\cdot|\cdot)$ denotes the hermitian form on $\mathcal{F}_+(\mathcal{H})$ defined by $(v|w) \equiv \langle v|\Gamma(\eta)w\rangle$ for $v, w \in \mathcal{F}_+(\mathcal{H})$. Furthermore, $\pi_B(\mathfrak{A}_B(\mathcal{H},\eta))\Omega$ is dense in $\mathcal{F}_+(\mathcal{H})$.

(ii) *There exists a self-adjoint unitary $\Gamma(\eta)$ on $\mathcal{F}_+(\mathcal{H})$, a dense subspace $\mathcal{D}$ of $\mathcal{F}_+(\mathcal{H})$ and an involutive representation $\pi_{B,0}$ of $(\mathcal{A}_B(\mathcal{H},\eta), \dagger)$ on the Krein space $(\mathcal{F}_+(\mathcal{H}), (\cdot|\cdot))$ such that*
$$\pi_{B,0}(a(f))\Omega = 0 \quad (\textit{for all } f \in \mathcal{H}), \quad \pi_{B,0}(\mathcal{A}_B(\mathcal{H},\eta))\Omega = \mathcal{D}$$
where $(\cdot|\cdot)$ is the hermitian form on $\mathcal{F}_+(\mathcal{H})$ defined by $(v|w) \equiv \langle v|\Gamma(\eta)w\rangle$ for $v, w \in \mathcal{F}_+(\mathcal{H})$.

(iii) *There exists a self-adjoint unitary $\Gamma(\eta)$ on $\mathcal{F}_-(\mathcal{H})$ and an involutive representation π_F of $(\mathfrak{A}_F(\mathcal{H},\eta), \dagger)$ on the Krein space $(\mathcal{F}_-(\mathcal{H}), (\cdot|\cdot))$ such that*
$$\pi_F(a(f))\Omega = 0 \quad (\textit{for all } f \in \mathcal{H})$$
where $(\cdot|\cdot)$ is the hermitian form on $\mathcal{F}_-(\mathcal{H})$ defined by $(v|w) \equiv \langle v|\Gamma(\eta)w\rangle$ for $v, w \in \mathcal{F}_-(\mathcal{H})$. Furthermore, $\pi_F(\mathfrak{A}_F(\mathcal{H},\eta))\Omega$ is dense in $\mathcal{F}_-(\mathcal{H})$.

Here topologies on $\mathcal{F}_+(\mathcal{H})$ and $\mathcal{F}_-(\mathcal{H})$ are taken as the norm topology induced by the inner product $\langle\cdot|\cdot\rangle$.

Proof. For (ii) and (iii), see Theorem 1.2 in [16]. We show (i). Let $\Gamma(\eta)$ denote the second quantization of η and let $\{W(f) : f \in \mathcal{H}\}$ denote the family of Weyl forms of CCRs on $\mathcal{F}_+(\mathcal{H})$. Since $\langle\Omega|W(f)\Omega\rangle = e^{-\|f\|^2/4}$ for each $f \in \mathcal{H}$ in Section 5.2.3 of [7], $(\Omega|\pi_B(W(f))\Omega) = \langle\Omega|\Gamma(\eta)\pi_B(W(f))\Omega\rangle = \langle\Omega|\pi_B(W(\eta(f))\Gamma(\eta)\Omega\rangle = \langle\Omega|\pi_B(W(\eta(f))\Omega\rangle = e^{-\|\eta f\|^2/4} = e^{-\|f\|^2/4}$. ∎

4.4 Equivalence

When $\eta = I$, the η-CCRs and the η-CARs coincide with ordinary CCRs and CARs, respectively. We classify algebras $\mathfrak{A}_B(\mathcal{H},\eta)$, $\mathcal{A}_B(\mathcal{H},\eta)$ and $\mathfrak{A}_F(\mathcal{H},\eta)$.

Proposition 1 *For two self-adjoint unitaries η and $\eta^{'}$ on $\mathcal{H}$, if there exists a unitary U on $\mathcal{H}$ such that $U\eta U^* = \eta^{'}$, then the following involutive isomorphisms hold:*

$$\mathfrak{A}_B(\mathcal{H},\eta) \cong \mathfrak{A}_B(\mathcal{H},\eta^{'}), \quad \mathcal{A}_B(\mathcal{H},\eta) \cong \mathcal{A}_B(\mathcal{H},\eta^{'}), \quad \mathfrak{A}_F(\mathcal{H},\eta) \cong \mathfrak{A}_F(\mathcal{H},\eta^{'}).$$

Proof. Assume that $\{a(f), a^\dagger(f) : f \in H\}$ is the set of canonical generators of $\mathcal{A}_B(\mathcal{H},\eta)$. Define $t(f) \equiv a(U^*f)$ and $t^\dagger(f) \equiv a^\dagger(U^*f)$ for $f \in \mathcal{H}$. Then we can verify that $\{t(f), t^\dagger(f) : f \in \mathcal{H}\}$ satisfy canonical relations of $\mathcal{A}_B(\mathcal{H},\eta^{'})$. Hence we obtain the embedding of $\mathcal{A}_B(\mathcal{H},\eta^{'})$ into $\mathcal{A}_B(\mathcal{H},\eta)$. Furthermore this mapping is surjective. Therefore $\mathcal{A}_B(\mathcal{H},\eta)$ and $\mathcal{A}_B(\mathcal{H},\eta^{'})$ are involutively isomorphic. In the same way, we can verify other cases. ∎

Corollary 2 *In Proposition 1, equivalences among algebras hold if*

$$(\mathrm{ind}_+(\eta), \mathrm{ind}_-(\eta)) = (\mathrm{ind}_+(\eta^{'}), \mathrm{ind}_-(\eta^{'}))$$

where $\mathrm{ind}_\pm(\eta) \equiv \dim\{x \in \mathcal{H} : \eta x = \pm x\}$.

Since the η-CCR for $\mathrm{rank}\eta = 2$ is unique up to involutive isomorphism (Example 3.3 in [16]), the inverse statement of Corollary 2 does not hold.

4.5 Algebra of FP ghosts

We reintroduce the algebra of FP ghosts in [2] as a Banach involutive algebra. As for the FP (anti) ghost fields in string theory, their mode-decomposed operators satisfy the abnormal anticommutation relations with the special structure owing to the hermiticity of the FP (anti) ghost fields as follows:

$$\begin{cases} c_0\bar{c}_0 + \bar{c}_0 c_0 = -I, \quad c_0^\dagger = c_0, \quad \bar{c}_0^\dagger = \bar{c}_0, \\ \\ c_m\bar{c}_n^\dagger + \bar{c}_n^\dagger c_m = c_m^\dagger \bar{c}_n + \bar{c}_n c_m^\dagger = -\delta_{m,n} I \quad (m,\, n = 1,\, 2\, \ldots) \end{cases} \tag{4.1}$$

and other anticommutation relations vanish. Define $\mathcal{FP}_0$ the involutive algebra generated by $\{c_n, \bar{c}_n : n \geq 0\}$. Define the self-adjoint unitary η on the Hilbert space $\mathcal{H} \equiv l_2(\mathbf{Z}_{\geq 0})$ by

$$\eta e_{2n} \equiv -e_{2n+1}, \quad \eta e_{2n+1} \equiv -e_{2n} \quad (n \geq 0) \tag{4.2}$$

where $\mathbf{Z}_{\geq 0} \equiv \{n \in \mathbf{Z} : n \geq 0\}$.

Lemma 3 *For η in (4.2), there exists an involutive embedding of $\mathcal{FP}_0$ into the η-CAR algebra $\mathfrak{A}_F(\mathcal{H},\eta)$.*

Proof. Let $\{a(f) : f \in \mathcal{H}\}$ denote the canonical generators of $\mathfrak{A}_F(\mathcal{H},\eta)$ and define $a_n \equiv a(e_n)$ for $n \geq 0$ where $\{e_n\}_{n\geq 0}$ denotes the standard basis of $\mathcal{H}$. Define the map φ from $\mathcal{FP}_0$ into $\mathfrak{A}_F(\mathcal{H},\eta)$ by

$$\varphi(c_0) \equiv 2^{-1/2}(a_0 + a_0^\dagger), \quad \varphi(\bar{c}) \equiv 2^{-1/2}(a_1 + a_1^\dagger),$$

$$\varphi(c_n) \equiv a_{2n}, \quad \varphi(\bar{c}_n) \equiv a_{2n+1} \quad (n \geq 1).$$

Then we can verify that φ is an involutive embedding. ∎

The completion $\mathcal{FP}$ of $\mathcal{FP}_0$ in $\mathfrak{A}_F(\mathcal{H},\eta)$ is the Krein C*-algebra of FP ghosts.

5 η-Cuntz algebra

In this section, we introduce η-Cuntz algebras as Krein C*-algebras.

5.1 Definition and equivalence

Let $\eta = (\eta_{ij})_{i,j=1}^N$ be a self-adjoint unitary in $U(N)$. Let $(\mathcal{O}_\eta^{(0)}, \dagger)$ denote the unital algebra $\mathcal{O}_\eta^{(0)}$ with an involution $\dagger$ generated by $s_1, \ldots, s_N$ which satisfies

$$s_i^\dagger s_j = \eta_{ij} I \quad (i,j = 1,\ldots,N), \quad \sum_{i,j=1}^N \eta_{ij} s_i s_j^\dagger = I. \tag{5.1}$$

Define the involutive automorphism α_η on $(\mathcal{O}_\eta^{(0)}, \dagger)$ by

$$\alpha_\eta(s_i) \equiv \sum_{j=1}^N \eta_{ji} s_j \quad (i = 1,\ldots,N). \tag{5.2}$$

Lemma 4 *The algebra $\mathcal{O}_\eta^{(0)}$ has a unique norm $\|\cdot\|$ such that $\|\alpha_\eta(x^\dagger)x\| = \|x\|^2$ for each $x \in \mathcal{O}_\eta^{(0)}$.*

Proof. Define the new involution $*$ on $\mathcal{O}_\eta^{(0)}$ by $x^* \equiv \alpha_\eta(x^\dagger)$ for $x \in \mathcal{O}_\eta^{(0)}$. Then we see that $(\mathcal{O}_\eta^{(0)}, *)$ is a dense involutive subalgebra of the Cuntz algebra $(\mathcal{O}_N, *)$ [8]. Since the norm $\|\cdot\|$ satisfies the C*-condition with respect

to $*$, the norm $\|\cdot\|$ is a unique C*-norm on the involutive algebra $(\mathcal{O}_\eta^{(0)}, *)$.
∎

The existence of $\mathcal{O}_\eta^{(0)}$ for any η is also verified in the proof of Lemma 4.

Definition 6 (i) *The completion $\mathcal{O}_\eta$ of $\mathcal{O}_\eta^{(0)}$ with respect to the norm in Lemma 4 is called the η-Cuntz algebra.*

(ii) *For nonnegative integers d, d' with $d + d' \geq 2$, the Banach involutive algebra $(\mathcal{O}_{d,d'}, \dagger)$ is called the pseudo-Cuntz algebra if $\mathcal{O}_{d,d'}$ is the η-Cuntz algebra for $\eta \equiv (\eta_{ij})_{i,j=1}^{d+d'} \in U(d+d')$ defined by*

$$(\eta_{ij})_{i,j=1}^{d+d'} \equiv \begin{pmatrix} I_d & 0 \\ 0 & -I_{d'} \end{pmatrix}. \tag{5.3}$$

Especially, $\mathcal{O}_{d,0} \cong \mathcal{O}_d$. In [2], we generalized the Cuntz algebra to the pseudo-Cuntz algebra without topology which is an involutive algebra of operators on an indefinite-metric space. In Definition 6 (ii), the pseudo-Cuntz algebra is redefined as a Banach involutive algebra without use of any representation.

The equivalence among $\{\mathcal{O}_\eta : \eta \in U(N),\ \eta^* = \eta\}$ is shown as follows:

Lemma 5 *If $\eta = \Lambda\eta'\Lambda^*$ for a unitary $\Lambda \in U(N)$, then $\mathcal{O}_\eta \cong \mathcal{O}_{\eta'}$.*

Proof. Let $s_1, \ldots, s_N$ denote the canonical generators of $\mathcal{O}_\eta$. Define elements $t_i \equiv \sum_{j=1}^N \Lambda_{ji}^* s_j$ in $\mathcal{O}_{\eta'}$ for $i = 1, \ldots, N$. Then we see that $t_1, \ldots, t_N$ satisfy the canonical relations of $\mathcal{O}_{\eta'}$. Therefore $\mathcal{O}_{\eta'}$ is embedded into $\mathcal{O}_\eta$. Furthermore this embedding is surjective. Hence the statement holds. ∎

Corollary 3 *For any self-adjoint element $\eta \in U(N)$, the algebra $\mathcal{O}_\eta$ is isomorphic to a certain pseudo-Cuntz algebra.*

Let $(\mathcal{A}, \dagger)$ denote the involutive algebra generated by s_1 and s_2 which satisfy the following:

$$s_1^\dagger s_2 = I, \quad s_1^\dagger s_1 = s_2^\dagger s_2 = 0, \quad s_1 s_2^\dagger + s_2 s_1^\dagger = I.$$

Then the involution $\dagger$ is indefinite. The algebra $(\mathcal{A}, \dagger)$ is densely, involutively embedded into $(\mathcal{O}_{1,1}, \dagger)$.

Example 4 Two algebras $\mathcal{O}_2$ and $\mathcal{O}_{1,1}$ are not involutive isomorphic because there is no nondegenerate involutive representation of $\mathcal{O}_{1,1}$ on Hilbert space. In the same reason, $\mathcal{O}_2$ and $\mathcal{O}_{0,2}$ are not involutive isomorphic. We let the following open problems: Whether are $\mathcal{O}_{2,1}$ and $\mathcal{O}_{1,2}$ equivalent or not? Whether are $\mathcal{O}_{1,1}$ and $\mathcal{O}_{0,2}$ equivalent or not? Classify all pseudo-Cuntz algebras.

For η in (5.3), ρ is the *canonical endomorphism* of $\mathcal{O}_{d,d'}$ if ρ is the map on $\mathcal{O}_{d,d'}$ defined by

$$\rho(x) \equiv \sum_{i=1}^{d+d'} \eta_{ii} s_i x s_i^{\dagger}.$$

We see that $\rho(x)\rho(y) = \rho(xy)$ for $x, y \in \mathcal{O}_{d,d'}$.

For η in (5.3), define $U(d,d') \equiv \{g \in GL_{d+d'}(\mathbf{C}) : g\eta g^* = \eta\}$. Then $U(d,d')$ is a group such that $g^* \in U(d,d')$ for each $g \in U(d,d')$ where $*$ denotes the hermite conjugate on $M_{d+d'}(\mathbf{C})$. We see that $U(d,d') = \{g \in GL_{d+d'}(\mathbf{C}) : g^*\eta g = \eta\}$. For $g \in U(d,d')$, define the involutive automorphism α_g of $\mathcal{O}_{d,d'}$ by

$$\alpha_g(s_i) \equiv \sum_{j=1}^{d+d'} g_{ji} s_j \quad (i = 1, \ldots, d+d').$$

Then α is an involutive action of $U(d,d')$ on $(\mathcal{O}_{d,d'}, \dagger)$. Especially, the $U(1)$-gauge action on $\mathcal{O}_{d,d'}$ is also an involutive action.

5.2 Involutive representation of $\mathcal{O}_{d,d'}$ on Krein space

According to Theorem 1, we consider involutive representations of pseudo-Cuntz algebras.

Corollary 4 *Let η be as in (5.3). Assume that $(\mathcal{H}, \pi, U)$ is the covariant representation of the C^*-dynamical system $(\mathcal{O}_{d+d'}, \mathbf{Z}_2, \alpha_\eta)$ for α_η in (5.2) and $N = d + d'$. Define the hermitian form $(\cdot|\cdot)$ on $\mathcal{H}$ by*

$$(v|w) \equiv \langle v|Uw\rangle \quad (v, w \in \mathcal{H}).$$

Then π is an involutive representation of $(\mathcal{O}_{d,d'}, \dagger)$ on the Krein space $(\mathcal{H}, (\cdot|\cdot))$.

We show examples.

Example 5 Let $(\mathcal{H},\pi)$ be a representation of $\mathcal{O}_2$ with a cyclic vector $\Omega \in \mathcal{H}$ which satisfies

$$\pi(s_1 s_2)\Omega = \Omega.$$

Such $(\mathcal{H},\pi)$ is $P(12)$ in [15] and it is unique up to unitary equivalence and irreducible. Define $\alpha \in \mathrm{Aut}\mathcal{O}_2$ and the new involution $\dagger$ on $\mathcal{O}_2$ by

$$\alpha(s_1) \equiv s_1, \quad \alpha(s_2) \equiv -s_2, \quad x^\dagger \equiv \alpha(x^*) \quad (x \in \mathcal{O}_2).$$

Then $\mathcal{O}_2$ becomes $\mathcal{O}_{1,1}$ by replacing the involution $*$ with $\dagger$. We construct an involutive representation of $(\mathcal{O}_{1,1},\dagger)$ from $(\mathcal{H},\pi)$ as follows. Since $(\mathcal{H},\pi\circ\alpha)$ is not equivalent to $(\mathcal{H},\pi)$, $(\mathcal{H},\pi)$ itself is not a covariant representation of $(\mathcal{O}_2,\mathbf{Z}_2,\alpha)$. From $(\mathcal{H},\pi)$, we obtain the representation $(\tilde{\mathcal{H}},\tilde{\pi})$ of $\mathcal{O}_2$ in (2.5). Define $V_1 \equiv \tilde{\pi}(\mathcal{O}_2)\Omega \otimes \mathbf{C}e_0$ and $V_2 \equiv \tilde{\pi}(\mathcal{O}_2)\Omega \otimes \mathbf{C}e_1$. We see that both $\tilde{\pi}|_{V_1}$ and $\tilde{\pi}|_{V_2}$ are irreducible representations of $\mathcal{O}_2$. On the other hand, $\mathcal{H}_+ \equiv \{v \otimes (e_0+e_1) : v \in \mathcal{H}\}$ and $\mathcal{H}_- \equiv \{v \otimes (e_0-e_1) : v \in \mathcal{H}\}$. Hence $\mathcal{H}_\pm$ is not invariant under the action of $\tilde{\pi}(\mathcal{O}_{1,1})$.

Example 6 Let $d \geq 1$ and $\eta_{ij} \equiv (-1)^{i-1}\delta_{ij}$ for $i,j = 1,2,\ldots,2d$. Define the $*$-automorphism of $\mathcal{O}_{2d}$ by

$$\alpha(s_i) \equiv \eta_{ii}s_i \quad (i = 1,2,3,\ldots,2d). \tag{5.4}$$

Then $\alpha^2 = id$. Define the new involution $\dagger$ on $\mathcal{O}_{2d}$ by $x^\dagger = \alpha(x^*)$ for $x \in \mathcal{O}_{2d}$. Then we see that

$$s_i^\dagger s_j = \eta_{ij}I \quad (i,j = 1,2,3,\ldots,2d), \qquad \sum_{i,j=1}^{2d} \eta_{ij}s_i s_j^\dagger = I.$$

Hence they generate $\mathcal{O}_{d,d}$. In this way, the replacement of involution on $\mathcal{O}_{2d}$ makes $\mathcal{O}_{2d}$ to $\mathcal{O}_{d,d}$.

Next, we construct an involutive representation of $\mathcal{O}_{d,d}$ as follows. Let $((\mathcal{H},\langle\cdot|\cdot\rangle),\pi)$ be a $*$-representation of $\mathcal{O}_{2d}$ with a cyclic unit vector $\Omega \in \mathcal{H}$ which satisfies

$$\pi(s_1)\Omega = \Omega.$$

By this assumption, we see that $(\mathcal{H},\pi)$ is irreducible. Let $\{1,\ldots,2d\}^* \equiv \bigcup_{k\geq 0}\{1,\ldots,2d\}^k$ where $\{1,\ldots,2d\}^0 \equiv \emptyset$. Define the function χ on $\{1,\ldots,2d\}^*$ by $\chi(\emptyset) \equiv 1$,

$$\chi(J) \equiv (-1)^{n_2(J)}$$

where $n_2(J) \equiv \#\{i \in \{1,\ldots,k\} : j_i \text{ is even}\}$ for $J = (j_1,\ldots,j_k)$. Define the subset $\Lambda \equiv \{(1), J \cup (i) : J \in \{1,2,\ldots,2d\}^*,\ 2 \leq i \leq 2d\}$ of multiindices and

let subsets $\Lambda_\pm$ of Λ by $\Lambda_\pm \equiv \{J \in \Lambda : \chi(J) = \pm 1\}$. Define two subspaces $\mathcal{H}_{\pm,0}$ of $\mathcal{H}$ by

$$\mathcal{H}_{\pm,0} \equiv \mathrm{Lin}\langle\{\pi(s_J)\Omega : J \in \Lambda_\pm\}\rangle$$

and let $\mathcal{H}_\pm$ denote their completions. Then $\{\pi(s_J)\Omega : J \in \Lambda\}$ is a complete orthonormal basis of $\mathcal{H}$ and $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. Define the unitary η on $\mathcal{H}$ by

$$\eta\pi(s_J)\Omega \equiv \chi(J)\pi(s_J)\Omega \quad (J \in \Lambda).$$

For α in (5.4), we can verify that $\pi\circ\alpha = \mathrm{Ad}\eta\circ\pi$. Let $e_1 \equiv \Omega$ and $e_i \equiv \pi(s_i)\Omega$ for $i = 2, 3, 4, \ldots, 2d$ and define $\{e_n \in \mathcal{H} : n \in \mathbf{N}\}$ recursively by

$$e_{4d(n-1)+i} \equiv \pi(s_i)e_{2n-1}, \quad e_{4dn+1-i} \equiv \pi(s_i)e_{2n} \quad (i = 1, 2, \ldots, 2d, , n \in \mathbf{N}).$$

Then $\{e_{2n-1} : n \in \mathbf{N}\} \subset \mathcal{H}_+$ and $\{e_{2n} : n \in \mathbf{N}\} \subset \mathcal{H}_-$. Define the new hermitian form $(\cdot|\cdot)$ on $\mathcal{H}$ by

$$(v|w) \equiv \langle v|\eta w\rangle \quad (v, w \in \mathcal{H}).$$

Then $(e_n|e_m) = \eta_{nm}$ for $n, m \in \mathbf{N}$. We see that $(\mathcal{H}, (\cdot|\cdot))$ is a Krein space with a fundamental decomposition $\mathcal{H}_+ \oplus \mathcal{H}_-$ and $((\mathcal{H}, (\cdot|\cdot)), \pi)$ is an involutive representation of the pseudo-Cuntz algebra $(\mathcal{O}_{d,d}, \dagger)$. This is just the example in Section 3 of [2].

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A Pauli's modified Schrödinger representation of abnormal canonical commutation relations

We review the Pauli's example in Section 3 of [28] as a covariant representation of the involutive algebra $\mathcal{A}$ generated by abnormal canonical commutation relations by modifying the Schrödinger representation. He strictly distinguished "hermitian operator" and "self-adjoint operator", and " $*$ " and "$\dagger$" as his notation and terminology. Remark that we change the notation and terminology from originals. Let $(\mathcal{A}, \dagger)$ denote the involutive algebra generated by a and $a^\dagger$ which satisfy (1.1). We see that $\mathcal{A} = \mathcal{A}_B(\mathbf{C}, -I)$ in Section 4.1. Define the involutive automorphism α of $(\mathcal{A}, \dagger)$ by $\alpha(a) \equiv -a$ and $\alpha(a^\dagger) \equiv -a^\dagger$. Define the new involution $*$ on $\mathcal{A}$ by $x^* \equiv \alpha(x^\dagger)$ for

$x \in \mathcal{A}$. Then we see that $aa^* - a^*a = I$. Define the covariant representation $(\mathcal{H}, \pi, R)$ of the involutive dynamical system $((\mathcal{A}, *), \mathbf{Z}_2, \alpha)$ by

$$\mathcal{H} \equiv L_2(\mathbf{R}), \quad \pi(a) \equiv 2^{-1/2}(\hat{p} - \sqrt{-1}\hat{q}),$$

$$R : \mathcal{H} \to \mathcal{H}; \quad (Rf)(q) \equiv f(-q) \quad (q \in \mathbf{R})$$

where $\hat{p} \equiv -\sqrt{-1}d/dq$ and let $\hat{q}$ denote the position operator on $\mathcal{H}$. Since operators $\hat{p}$ and $\hat{q}$ are extended to self-adjoint operators on $\mathcal{H}$, the adjoint $\pi(a)^*$ of $\pi(a)$ with respect to the standard inner product on $L_2(\mathbf{R})$ is given by $\pi(a)^* = 2^{-1/2}(\hat{p} + \sqrt{-1}\hat{q})$. We see that $\pi \circ \alpha = \mathrm{Ad}R \circ \pi$. Hence the algebra $(\mathcal{A}, \dagger)$ is involutively represented on the Krein space $(\mathcal{H}, (\cdot|\cdot))$ where the indefinite metric $(\cdot|\cdot)$ on $\mathcal{H}$ is defined by

$$(f|g) \equiv \int_{\mathbf{R}} \overline{f(q)}\, \{Rg\}(q)\, dq = \int_{\mathbf{R}} \overline{f(q)}\, g(-q)\, dq \quad (f, g \in \mathcal{H}).$$

Then the adjoint $\pi(a)^\dagger$ of $\pi(a)$ with respect to $(\cdot|\cdot)$ is given by

$$\pi(a)^\dagger = 2^{-1/2}(-\hat{p} - \sqrt{-1}\hat{q}) = -\pi(a)^*.$$

Furthermore $\hat{p}^\dagger = -\hat{p}$ and $\hat{q}^\dagger = -\hat{q}$. For $\mathcal{H}_\pm \equiv \{v \in \mathcal{H} : Rv = \pm v\}$, we see that $\mathcal{H}_+$ (*resp.* $\mathcal{H}_-$) is the space of all even (*resp.* odd) functions in $\mathcal{H}$. In consequence, the replacement of involution on the Schrödinger representation gives an involutive representation of abnormal canonical commutation relations on the Krein space.

B A model with indefinite metric

It is known that there are various simple models associated with indefinite-metric quantum field theory [3, 13, 27]. Araki treated a system of a boson and an abnormal boson [3]. Nakanishi treated a system of a fermion, an abnormal fermion and a countably infinite family of bosons in the Lee model (Section 12, [27]. See also [10]). They computed eigenvalues of Hamiltonians which are apparently self-adjoint, and showed that there exist complex eigenvalues which are not real under certain conditions. Such Hamiltonian is treated as "pseudo-Hermitian operator" in [32].

In order to effectively explain how the indefinite-metric space appears and why non-real valued eigenvalues are derived, we introduce a simpler model as a *virtual system* of transformation among a particle A to a particle B

$$A \leftrightarrows B.$$

Assume that A is a boson ($= b$) or a fermion ($= f$), and B is an abnormal boson ($= \bar{b}$) or an abnormal fermion ($= \bar{f}$). Hence there are four combinations of the choice of particles A and B as follows:

$$(A, B) = (b, \bar{b}), (b, \bar{f}), (f, \bar{b}), (f, \bar{f}). \tag{B.1}$$

For every combination, we define the (common) Hamiltonian H by

$$H \equiv m_A a_A^\dagger a_A - m_B a_B^\dagger a_B + g a_A^\dagger a_B + \bar{g} a_B^\dagger a_A \tag{B.2}$$

where $m_X, a_X^\dagger, a_X$ denote the mass, the creation and the annihilation operator of $X = A, B$ and $g \in \mathbf{C}$ is their coupling constant. We assume that $a_A a_B = a_B a_A$ and $a_A^\dagger a_B = a_B a_A^\dagger$. Let Ω denote the common vacuum vector such that $a_A \Omega = a_B \Omega = 0$. Then we see that the hermitian form $(\cdot|\cdot)$ on the state space such that H is self-adjoint and Ω is normalized, satisfies the following:

$$(\Omega|\Omega) = (a_A^\dagger \Omega | a_A^\dagger \Omega) = 1, \quad (a_B^\dagger \Omega | a_B^\dagger \Omega) = -1 \tag{B.3}$$

and $\Omega, a_A^\dagger \Omega, a_B^\dagger \Omega$ are mutually orthogonal.

Proposition 2 *Let $V \equiv \mathrm{Lin}\langle \{a_A^\dagger \Omega, a_B^\dagger \Omega\} \rangle$ and let H be as in (B.2). Then $HV \subset V$ and the eigenvalue of H on V is the following:*

(i) *If $|m_A - m_B| > 2|g|$, then H has two different real eigenvalues. The norm of one of these two eigenvectors is negative.*

(ii) *If $|m_A - m_B| = 2|g|$, then H has one real eigenvalue. The associated eigenvector is neutral, that is, zero-norm.*

(iii) *If $|m_A - m_B| < 2|g|$, then H has two different eigenvalues which are not real.*

Proof. For each case in (B.1), the first statement is easily verified and the matrix representation of H with respect to $a_A^\dagger \Omega$ and $a_B^\dagger \Omega$ is

$$H|_V = \begin{pmatrix} m_A & -g \\ \bar{g} & m_B \end{pmatrix}. \tag{B.4}$$

By the discriminant of the eigenequation of this matrix, the statement (i),(ii) and (iii) about the properties of eigenvectors hold. The norm of the eigenvector with respect to the indefinite metric $(\cdot|\cdot)$ is verified by direct computation. ∎

The operator H in (B.2) is hermite on the state space with the indefinite metric $(\cdot|\cdot)$ which satisfies (B.3). However the matrix in (B.4) is not hermite where (B.4) is obtained from the comparison of the left side and the right side of the eigenequation. The result of Proposition 2 is similar to that of the Lee model in [27]. For our model, it is not necessary to use the topology of indefinite-metric space and operators on it because $\dim V < \infty$.

We consider the above model as a representation theory of involutive algebra. Let $(\mathcal{A}_X, \dagger)$ denote the involutive algebra generated by a_X and let $\mathcal{H}_X \equiv \mathcal{F}_+(\mathbf{C}, \eta)$ or $\mathcal{F}_-(\mathbf{C}, \eta)$ for $X = A, B$, respectively. If A is a boson, then $\mathcal{H}_A \cong l_2(\mathbf{N})$ and $\eta = I$. If A is a fermion, then $\mathcal{H}_A \cong \mathbf{C}^2$ and $\eta = I$. The total algebra is $\mathcal{A}_A \otimes \mathcal{A}_B$ and the representation space is $\mathcal{H}_A \otimes \mathcal{H}_B$. The Hamiltonian H belongs to $\mathcal{A}_A \otimes \mathcal{A}_B$. Define the involutive automorphism α of $\mathcal{A}_A \otimes \mathcal{A}_B$ by $\alpha(a_A) \equiv a_A, \alpha(a_A^\dagger) \equiv a_A^\dagger, \alpha(a_B) \equiv -a_B$ and $\alpha(a_B^\dagger) \equiv -a_B^\dagger$ where we identify a_A and a_B with $a_A \otimes I$ and $I \otimes a_B$, respectively. For the new involution $x^* \equiv \alpha(x^\dagger)$, we obtain that

$$H = m_A a_A^* a_A + m_B a_B^* a_B + g a_A^* a_B - \bar{g} a_B^* a_A.$$

Then we see that H is not self-adjoint with respect to $*$ on the positive definite-metric space. Since the eigenvalue of H is independent in the choice of hermitian form, the result in (2) holds.

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